ACCURACY

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ABSTRACT

One of the difficulties in the past relating to product accuracy has been the absence of agreement on the definitions of accuracy and the appropriate application of these definitions. There has been an increasing need for DMA to better explain accuracy to product consumers, other government agencies and contractors supporting DMA. Recent discussions within DMA has led to a consensus of opinion on the proper definitions. This paper states the definitions and expands them to a meaningful mathematical formulation.

This report defines accuracy and suggests mathematical tools to propagate the effects of errors to a generic product via standard covariance error propagation and/or sample statistics arising from comparisons to diagnostic control.

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I. Introduction

The purpose of this report is to standardize both the definitions and mathematics of absolute and point to point (relative) accuracy. It is understood that these are the official statistics for stating product accuracies and for specifying hardware/software requirements when these specifications are stated in terms of ground position accuracies. The emphasis of this report is on the development of the theory which defines accuracy. The application of that theory to any given product is not presented.

Absolute accuracy is defined as the statistics which gives the uncertainty of a point with respect to the datum required by a product specification. This definition implies that the effects of all error sources, both random and systematic, must be considered. Absolute accuracy is stated in terms of two components, a horizontal component and a vertical component. The horizontal absolute accuracy associated with a product is stated as a circular error, CE, such that 90 percent of all positions depicted by that product have a horizontal error with magnitude less than CE. Likewise the absolute vertical accuracy associated with a product is stated as a linear error, LE, such that 90 percent of all elevations depicted by the product have an error with magnitude less than LE.

Relative accuracy is that statistic which gives the uncertainty between the positions of two points after the effects of all errors common to both points have been removed. Relative accuracy is also called point to point accuracy. Relative accuracy is seen to be independent of product datum in that it is defined as the error in the components of the vector between the two points; but is still stated in terms of a horizontal component and a vertical component. As in the case with absolute accuracy, the horizontal uncertainty is stated as a CE and the vertical error is stated as a LE.

Point positions derived from measurements of photographic images are usually referenced to an earth fixed Cartesian coordinate system. A variance-covariance matrix defining the uncertainty of this computed position relative to this coordinate system is determined by standard error propagation techniques utilizing a apriori estimates of errors associated with the computational parameters. The a apriori estimates of the errors associated with these computational parameters are usually in the form of a variance-covariance matrix and includes all of the covariances resulting from the correlation of the parameters. The parameter variance-covariance matrices used to assess product accuracies result from; (1) statistics accumulated from redundant observations of the parameters or (2) statistics propagated through computations required to determine the parameters from redundant indirect observations. An example of such computations is those required to accomplish least squares triangulation to update exposure station positions and camera attitudes.

A primary goal of any evaluation scheme should be the construction of the variance-covariance matrix associated with any position depicted in the product. The generation of such matrices will likely utilize standard error propagation techniques and/or sample statistics resulting from the comparison

of positions extracted from the product to their known positions. Such points are referred to as diagnostic points. Ultimately the success of any evaluation method depends on its ability to approximate these variance-covariance matrices.

The variance-covariance matrix relating the errors of two geographic positions will be defined. This is followed by a summary of methods used in the determination of this matrix in various circumstances. Finally the computation of the absolute CE and LE and the relative point to point CE and LE is presented.

To define a covariance matrix consider two vectors, denoted by U and V, whose components are random variables. The cross-covariance of the two vectors is defined by

$$E[(U - E[U])(V - E[V])^T] \qquad (I.1)$$

where E is the expectation of the random variable and is defined as the sum of all values the random variable may take, each weighted by the probability of its occurrence. The covariance of U is when $U = V$.

Suppose that the geographic position of two points, and their cross-covariance matrix has been determined. Let the two positions be denoted by (ϕ_1, λ_1, h_1) and (ϕ_2, λ_2, h_2). Let their cross-covariance matrix be denoted by Q such that

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \qquad (I.2)$$

where

$$Q_{ii} = \begin{bmatrix} \sigma_{\phi_i}^2 & \sigma_{\phi_i\lambda_i} & \sigma_{\phi_i h_i} \\ \sigma_{\phi_i\lambda_i} & \sigma_{\lambda_i}^2 & \sigma_{\lambda_i h_i} \\ \sigma_{\phi_i h_i} & \sigma_{\lambda_i h_i} & \sigma_{h_i}^2 \end{bmatrix} \qquad (I.3)$$

where $\sigma_{\phi_i}^2$ is the variance of ϕ_i, etc.,

$\sigma_{\phi_i\lambda_i}$ is the covariance of ϕ_i and λ_i ,etc.

and

$$Q_{12} = \begin{bmatrix} \sigma_{\phi_1\phi_2} & \sigma_{\phi_1\lambda_2} & \sigma_{\phi_1 h_2} \\ \sigma_{\lambda_1\phi_2} & \sigma_{\lambda_1\lambda_2} & \sigma_{\lambda_1 h_2} \\ \sigma_{h_1\phi_2} & \sigma_{h_1\lambda_2} & \sigma_{h_1 h_2} \end{bmatrix} . \quad (I.4)$$

Methods for the determination of the cross-covariance matrix Q will be considered. These methods ,intended as guidelines only, are somewhat generalized in the sense that they are not presented in terms of any one product. Two methods are presented; the first based on the statistics output from triangulation; the second based on a comparison of positions sampled from the product to known or diagnostic positions.

II. Error Propagation Relating to Triangulation

First consider the case involving triangulation. It is not within the scope of this report to present an exhaustive development of triangulation mathematics. Hopefully enough for clarity and understanding is presented.

The condition equations are assumed to be of the form

$$A(L + V) + B\Delta = D \quad (II.1)$$

where A and B are coefficient matrices,

D is a vector of constants,

L is a vector of observations,

V is a vector of residuals and

Δ is a vector of parameters usually referred to as the state vector.

In addition define Q_{LL} as the covariance matrix associated with the observational vector L and define W as the observational weight matrix, that is,

$$W = Q_{LL}^{-1} \quad (II.2)$$

A few words relative to the observations and state vector regarding their respective weights are in order. Assume that the unknown state vector, , has an initial value that results from an observational reduction process and thus can be treated as part of the observations, L . Thus any theoretical error propagation scheme used to estimate triangulation output accuracies depend heavily on a priori covariances associated with the observations or associated with parameters treated as observations. The covariance matrices resulting from triangulation are considered acceptable if a reference variance computed from the residuals is believable. Define this reference variance as

$$\sigma_o^2 = \frac{V^T W V}{R} \qquad (II.3)$$

where R is the degrees of freedom associated with the least squares adjustment. Since the weight matrix is the inverse of the observational covariance matrix, the reference variance is in variance units and will be near unity in value. In fact σ_o^2 is sometimes referred to as the the unit variance. If the unit variance is not close to unity it becomes difficult to give much credibility to the subsequent error propagation.

Rearrange the condition equations so that the form is

$$AV + B\Delta = F \qquad (II.4)$$

with

$$F = D - AL. \qquad (II.5)$$

The least squares solution is defined as that solution which minimizes the function

$$\Phi = V^T W V - 2K^T(AV + B\Delta - F) \qquad (II.6)$$

with respect to V and Δ. The vector K is the Lagrange multipliers which accomplishes this minimization. Therefore to minimize Φ

$$\partial\Phi/\partial V = 0 \quad \text{and} \quad \partial\Phi/\partial\Delta = 0 \qquad (II.7)$$

must be satisfied. Thus

$$\partial\Phi/\partial V = 2V^T W - 2K^T A = 0 \qquad (II.8)$$

and

$$\partial\Phi/\partial\Delta = -2K^T B = 0 \qquad (II.9)$$

along with the condition equations forms the system of equations

$$WV - A^T K = 0, \qquad (II.10)$$

$$AV + B\Delta = F \quad \text{and} \qquad (II.11)$$

$$B^T K = 0 \qquad (II.12)$$

which must be solved for V, K and Δ. It can be shown that the solution is given by

$$V = Q_{LL} A^T K , \tag{II.13}$$

$$K = (A Q_{LL} A^T)^{-1} (F - B\Delta) \quad \text{and} \tag{II.14}$$

$$\Delta = [B^T (A Q_{LL} A^T)^{-1} B]^{-1} B^T (A Q_{LL} A^T)^{-1} F . \tag{II.15}$$

Let

$$N = B^T (A Q_{LL} A^T)^{-1} B \tag{II.16}$$

and

$$T = B^T (A Q_{LL} A^T)^{-1} F . \tag{II.17}$$

The normal equations can be written as

$$N\Delta = T \tag{II.18}$$

so that

$$\Delta = N^{-1} T . \tag{II.19}$$

The covariance matrix associated with the parameter Δ is determined by using the covariance propagation rule

$$Q_{\Delta\Delta} = J_{\Delta L} Q_{LL} J_{\Delta L}^T \tag{II.20}$$

where

$$J_{\Delta L} = \partial\Delta / \partial L . \tag{II.21}$$

Since

$$\Delta = N^{-1} B^T (A Q_{LL} A^T)^{-1} (D - AL) \tag{II.22}$$

it follows that

$$J_{\Delta L} = N^{-1} B^T (A Q_{LL} A^T)^{-1} (-A) \tag{II.23}$$

and

$$Q_{\Delta\Delta} = -N^{-1} B^T (A Q_{LL} A^T)^{-1} A Q_{LL} [-N^{-1} B^T (A Q_{LL} A^T)^{-1} A]^T \tag{II.24}$$

which simplifies to

$$Q_{\Delta\Delta} = N^{-1}. \tag{II.25}$$

It is often true that not all of the parameters in the state vector, Δ, are used for the development of a product. For example the state vector may include both ground positions and sensor related parameters. Some products may be developed using only the ground positions, while others may also utilize the sensor parameters. To understand this situation suppose that the state vector can be written as

$$\Delta = \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} \tag{II.26}$$

and the corresponding condition equations become

$$AV + \dot{B}\dot{\Delta} + \ddot{B}\ddot{\Delta} = F \tag{II.27}$$

which can be written as

$$AV + \begin{bmatrix} \dot{B} & \ddot{B} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = F \tag{II.28}$$

As before the normal equations, with

$$B = \begin{bmatrix} \dot{B} & \ddot{B} \end{bmatrix} \tag{II.29}$$

have the form

$$\begin{bmatrix} \dot{B}^T \\ \ddot{B}^T \end{bmatrix} (AQ_{LL}A^T)^{-1} \begin{bmatrix} \dot{B} & \ddot{B} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{B}^T \\ \ddot{B}^T \end{bmatrix} (AQ_{LL}A^T)^{-1} F. \tag{II.30}$$

To simplify the notation let

$$W_e = (AQ_{LL}A^T)^{-1} \tag{II.31}$$

thus

$$\begin{bmatrix} \dot{B}^T W_e \dot{B} & \dot{B}^T W_e \ddot{B} \\ \ddot{B}^T W_e \dot{B} & \ddot{B}^T W_e \ddot{B} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{B}^T W_e F \\ \ddot{B}^T W_e F \end{bmatrix}. \tag{II.32}$$

Let

$$\dot{N} = \dot{B}^T W_e \dot{B}, \tag{II.33}$$

$$\bar{N} = \dot{B}^T W_e \ddot{B}, \tag{II.34}$$

$$\ddot{N} = \ddot{B}^T W_e \ddot{B}, \tag{II.35}$$

$$\dot{T} = \dot{B}^T W_e F \quad \text{and} \tag{II.36}$$

$$\ddot{T} = \ddot{B}^T W_e F, \tag{II.37}$$

then the normal equations are

$$\begin{bmatrix} \dot{N} & \bar{N} \\ \bar{N}^T & \ddot{N} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{T} \\ \ddot{T} \end{bmatrix}. \tag{II.38}$$

Next solve for $\dot{\Delta}$ and $\ddot{\Delta}$ and determine $Q_{\dot{\Delta}\dot{\Delta}}$ and $Q_{\ddot{\Delta}\ddot{\Delta}}$, their respective covariance matrices. The normal equation can be written as

$$\dot{N}\dot{\Delta} + \bar{N}\ddot{\Delta} = \dot{T} \tag{II.39}$$

and

$$\bar{N}^T\dot{\Delta} + \ddot{N}\ddot{\Delta} = \ddot{T}. \tag{II.40}$$

Equation (II.39) yields

$$\dot{\Delta} = \dot{N}^{-1}(\dot{T} - \bar{N}\ddot{\Delta}) \tag{II.41}$$

which when substituted into equation (II.40) yields

$$\bar{N}^T\dot{N}^{-1}(\dot{T} - \bar{N}\ddot{\Delta}) + \ddot{N}\ddot{\Delta} = \ddot{T} \tag{II.42}$$

which reduces to

$$\ddot{\Delta} = (\ddot{N} - \bar{N}^T\dot{N}^{-1}\bar{N})^{-1}(\ddot{B}^T - \bar{N}^T\dot{N}^{-1}\dot{B}^T)W_e(D - AL). \tag{II.43}$$

The covariance propagation rule states that

$$Q_{\ddot{\Delta}\ddot{\Delta}} = [\partial\ddot{\Delta}/\partial L]\, Q_{LL}\, [\partial\ddot{\Delta}/\partial L]^T \tag{II.44}$$

where

$$\partial\ddot{\Delta}/\partial L = -(\ddot{N} - \bar{N}^T\dot{N}^{-1}\bar{N})^{-1}(\ddot{B}^T - \bar{N}^T\dot{N}^{-1}\dot{B}^T)W_e A \quad (II.45)$$

thus

$$Q_{\ddot{\Delta}\ddot{\Delta}} = (\ddot{N} - \bar{N}^T\dot{N}^{-1}\bar{N})^{-1}. \quad (II.46)$$

Likewise solve for $\ddot{\Delta}$ using equation (II.40), that is,

$$\ddot{\Delta} = \ddot{N}^{-1}(\ddot{T} - \bar{N}^T\dot{\Delta}) \quad (II.47)$$

which when substituted into equation (II.39) becomes

$$\dot{N}\dot{\Delta} + \bar{N}\ddot{N}^{-1}(\ddot{T} - \bar{N}^T\dot{\Delta}) = \dot{T} \quad (II.48)$$

which reduces to

$$\dot{\Delta} = (\dot{N} - \bar{N}\ddot{N}^{-1}\bar{N}^T)^{-1}(\dot{B}^T - \bar{N}\ddot{N}^{-1}\ddot{B}^T)W_e F. \quad (II.49)$$

The covariance matrix associated with $\dot{\Delta}$ is given by

$$Q_{\dot{\Delta}\dot{\Delta}} = [\partial\dot{\Delta}/\partial L]\, Q_{LL}\, [\partial\dot{\Delta}/\partial L]^T \quad (II.50)$$

where

$$\partial\dot{\Delta}/\partial L = -(\dot{N} - \bar{N}\ddot{N}^{-1}\bar{N}^T)^{-1}(\dot{B}^T - \bar{N}\ddot{N}^{-1}\ddot{B}^T)W_e A \quad (II.51)$$

thus

$$Q_{\dot{\Delta}\dot{\Delta}} = (\dot{N} - \bar{N}\ddot{N}^{-1}\bar{N}^T)^{-1}(\dot{B}^T - \bar{N}\ddot{N}^{-1}\ddot{B}^T)W_e A Q_{LL} A^T \times [(\dot{N} - \bar{N}\ddot{N}^{-1}\bar{N}^T)^{-1}(\dot{B}^T - \bar{N}\ddot{N}^{-1}\ddot{B}^T)W_e]^T \quad (II.52)$$

which simplifies to

$$Q_{\dot{\Delta}\dot{\Delta}} = (\dot{N} - \bar{N}\ddot{N}^{-1}\bar{N}^T)^{-1}. \quad (II.53)$$

It will now be shown that these expressions for $Q_{\dot{\Delta}\dot{\Delta}}$ and $Q_{\ddot{\Delta}\ddot{\Delta}}$ correspond to the partitions of N^{-1}. Assume that $\Delta = MT$, that is

$$\begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{M} & \bar{M} \\ \bar{M}^T & \ddot{M} \end{bmatrix} \begin{bmatrix} \dot{T} \\ \ddot{T} \end{bmatrix} \quad (II.54)$$

or

$$M = N^{-1} \tag{II.55}$$

which means that

$$\begin{bmatrix} \dot{N} & \bar{N} \\ \bar{N}^T & \ddot{N} \end{bmatrix} \begin{bmatrix} \dot{M} & \bar{M} \\ \bar{M}^T & \ddot{M} \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} \tag{II.56}$$

which when expanded gives the four equations

$$\dot{N}\dot{M} + \bar{N}\bar{M}^T = I, \tag{II.57}$$

$$\dot{N}\bar{M} + \bar{N}\ddot{M} = O, \tag{II.58}$$

$$\bar{N}^T\dot{M} + \ddot{N}\bar{M}^T = O \quad \text{and} \tag{II.59}$$

$$\bar{N}^T\bar{M} + \dddot{N}\dddot{M} = I. \tag{II.60}$$

Equation (II.59) can be rearranged so that

$$\bar{M}^T = -\ddot{N}^{-1}\bar{N}^T\dot{M} \tag{II.61}$$

which when substituted into equation (II.57) gives

$$\dot{N}\dot{M} + \bar{N}(-\ddot{N}^{-1}\bar{N}^T\dot{M}) = I \tag{II.62}$$

which, solving for $\dot{M}$, gives

$$\dot{M} = (\dot{N} - \bar{N}\ddot{N}^{-1}\bar{N}^T)^{-1} \tag{II.63}$$

Likewise, equation (II.58), when rearranged, gives

$$\bar{M} = -\dot{N}^{-1}\bar{N}\ddot{M} \tag{II.64}$$

which when substituted into equation (II.60) and solving for $\ddot{M}$ yields

$$\ddot{M} = (\ddot{N} - \bar{N}^T\dot{N}^{-1}\bar{N})^{-1}. \tag{II.65}$$

Thus it has been shown that

$$Q_{\dot{\Delta}\dot{\Delta}} = \dot{M} \tag{II.66}$$

and

$$Q_{\ddot{\Delta}\ddot{\Delta}} = \ddot{M}. \tag{II.67}$$

A typical method of reducing the dimension of the matrix to be inverted is to "fold" the normal equations. This is accomplished by eliminating some of the parameters from the state vector. Assume that the normal equations are partitioned as before, that is

$$\begin{bmatrix} \dot{N} & \bar{N} \\ \bar{N}^T & \ddot{N} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{T} \\ \ddot{T} \end{bmatrix} \tag{II.68}$$

which when expanded gives the two equations

$$\dot{N}\dot{\Delta} + \bar{N}\ddot{\Delta} = \dot{T} \tag{II.69}$$

and

$$\bar{N}^T\dot{\Delta} + \ddot{N}\ddot{\Delta} = \ddot{T}. \tag{II.70}$$

"Folding" is accomplished by solving equation (II.70) for $\ddot{\Delta}$ and then substituting the resulting expression into equation (II.69) and solving for $\dot{\Delta}$. The resulting expression is called the folded normal equations, that is,

$$\bar{N}^T\dot{\Delta} + \ddot{N}\ddot{\Delta} = \ddot{T} \tag{II.71}$$

implies that

$$\ddot{\Delta} = \ddot{N}^{-1}(\ddot{T} - \bar{N}^T\dot{\Delta}) \tag{II.72}$$

which upon substitution into equation (II.69) yields

$$\dot{N}\dot{\Delta} + \bar{N}\ddot{N}^{-1}(\ddot{T} - \bar{N}^T\dot{\Delta}) = \dot{T} \tag{II.73}$$

which reduces to

$$(\dot{N} - \bar{N}\ddot{N}^{-1}\bar{N}^T)\dot{\Delta} = \dot{T} - \bar{N}\ddot{N}^{-1}\ddot{T} \qquad (II.74)$$

and since

$$\dot{N} - \bar{N}\ddot{N}^{-1}\bar{N}^T = \dot{M}^{-1} \qquad (II.75)$$

the folded normal equations are

$$\dot{M}^{-1}\dot{\Delta} = \dot{T} - \bar{N}\ddot{N}^{-1}\ddot{T}. \qquad (II.76)$$

It will now be shown that the covariance matrix, $Q_{\dot{\Delta}\dot{\Delta}}$, is given by the same expression as in the unfolded case, that is,

$$Q_{\dot{\Delta}\dot{\Delta}} = \dot{M}. \qquad (II.77)$$

Since

$$\dot{\Delta} = \dot{M}(\dot{T} - \bar{N}\ddot{N}^{-1}\ddot{T}) \qquad (II.78)$$

substitution for $\dot{T}$ and $\ddot{T}$ gives

$$\dot{\Delta} = \dot{M}(\dot{B}^T - \bar{N}\ddot{N}^{-1}\ddot{B}^T)W_e(D - AL) \qquad (II.79)$$

Using the covariance propagation rule and the fact that

$$\partial\dot{\Delta}/\partial L = -\dot{M}(\dot{B}^T - \bar{N}\ddot{N}^{-1}\ddot{B}^T)W_e A \qquad (II.80)$$

yields

$$Q_{\dot{\Delta}\dot{\Delta}} = \dot{M}(\dot{B}^T - \bar{N}\ddot{N}^{-1}\ddot{B}^T)W_e A Q_{LL} A^T W_e \times (\dot{B}^T - \bar{N}\ddot{N}^{-1}\ddot{B}^T)^T \dot{M} \qquad (II.81)$$

which reduces to

$$Q_{\dot{\Delta}\dot{\Delta}} = \dot{M} \qquad (II.82)$$

which is also the result in the unfolded case.

With the parameters $\dot{\Delta}$ now known, along with $Q_{\dot{\Delta}\dot{\Delta}}$ it is possible to determine $\ddot{\Delta}$ and $Q_{\ddot{\Delta}\ddot{\Delta}}$. Consider

$$\ddot{\Delta} = \ddot{N}^{-1}(\ddot{T} - \bar{N}^T\dot{\Delta}) \qquad (II.83)$$

and substitution for $\ddot{T}$ yields

$$\ddot{\Delta} = \ddot{N}^{-1}[\ddot{B}^T W_e(D - AL) - \bar{N}^T\dot{\Delta}] \qquad (II.84)$$

thus $\ddot{\Delta}$ is a function of both the parameters $\dot{\Delta}$ and the observations L and the covariance propagation rule is

$$Q_{\ddot{\Delta}\ddot{\Delta}} = \begin{bmatrix} \partial\ddot{\Delta}/\partial L & \partial\ddot{\Delta}/\partial\dot{\Delta} \end{bmatrix} \begin{bmatrix} Q_{LL} & 0 \\ 0 & Q_{\dot{\Delta}\dot{\Delta}} \end{bmatrix} \begin{bmatrix} (\partial\ddot{\Delta}/\partial L)^T \\ (\partial\ddot{\Delta}/\partial\dot{\Delta})^T \end{bmatrix} \qquad (II.85)$$

where

$$\partial\ddot{\Delta}/\partial L = -\ddot{N}^{-1}\ddot{B}^T W_e A \qquad (II.86)$$

and

$$\partial\ddot{\Delta}/\partial\dot{\Delta} = -\ddot{N}^{-1}\bar{N}^T. \qquad (II.87)$$

Therefore

$$Q_{\ddot{\Delta}\ddot{\Delta}} = \begin{bmatrix} -\ddot{N}^{-1}\ddot{B}^T W_e A & -\ddot{N}^{-1}\bar{N}^T \end{bmatrix} \begin{bmatrix} Q_{LL} & 0 \\ 0 & Q_{\dot{\Delta}\dot{\Delta}} \end{bmatrix} \times \begin{bmatrix} -A^T W_e \ddot{B}\ddot{N}^{-1} \\ -\bar{N}\ddot{N}^{-1} \end{bmatrix} \qquad (II.88)$$

which reduces to

$$Q_{\ddot{\Delta}\ddot{\Delta}} = \ddot{N}^{-1} + \ddot{N}^{-1}\bar{N}^{T}\dot{M}\bar{N}\ddot{N}^{-1} \qquad (II.89)$$

with the substitution

$$Q_{\dot{\Delta}\dot{\Delta}} = \dot{M}. \qquad (II.90)$$

Equation (II.89) can be used to compute $Q_{\ddot{\Delta}\ddot{\Delta}}$ when the normal equations have been folded. It will now be shown that this expression for $Q_{\ddot{\Delta}\ddot{\Delta}}$ is equivalent to that obtained in the unfolded case, namely $\dot{M}$.

Since

$$\dot{N}\bar{M} + \bar{N}\ddot{M} = 0 \qquad (II.91)$$

and

$$\bar{N}^{T}\dot{M} + \ddot{N}\bar{M}^{T} = 0 \qquad (II.92)$$

it follows that

$$\dot{M}\bar{N}\ddot{N}^{-1} = \dot{N}^{-1}\bar{N}\ddot{M} \qquad (II.93)$$

thus

$$Q_{\ddot{\Delta}\ddot{\Delta}} = \ddot{N}^{-1} + \ddot{N}^{-1}\bar{N}^{T}\dot{N}^{-1}\bar{N}\ddot{M}. \qquad (II.94)$$

Since

$$\ddot{M} = (\ddot{N} - \bar{N}^{T}\dot{N}^{-1}\bar{N})^{-1} \qquad (II.95)$$

it follows that

$$\bar{N}^{T}\dot{N}^{-1}\bar{N} = \ddot{N} - \ddot{M}^{-1} \qquad (II.96)$$

thus

$$Q_{\ddot{\Delta}\ddot{\Delta}} = \ddot{N}^{-1} + \ddot{N}^{-1}(\ddot{N} - \ddot{M}^{-1})\ddot{M} \qquad (II.97)$$

which reduces to

$$Q_{\ddot{\Delta}\ddot{\Delta}} = \ddot{M} \qquad (II.98)$$

as in the unfolded case.

III. Application of Triangulation Output

For the purpose of applying this information to product evaluation it is assumed that the vector $\ddot{\Delta}$ has as its components ground location coordinates which are to be used as diagnostic control points by the production organization. The covariance matrices for $\dot{\Delta}$ and $\ddot{\Delta}$ are given by $Q_{\dot{\Delta}\dot{\Delta}}$ and $Q_{\ddot{\Delta}\ddot{\Delta}}$ respectively.

It should be noted that the organization generating some specific product may not use sensor parameters, that is, the organization will not be supplied with the vector $\dot{\Delta}$. Those organizations that require sensor parameters have two possible sources, either the output of triangulation in the form of a posteriori parameters or the output of some observational process in the form of a priori parameters. In either instance the same notation is used herein.

The possible sources of the ground positions used by the production organizations are;

- the output from triangulation, although it is unlikely that the diagnostic control points are part of the triangulation process,
- the output from a derivation utilizing either the a priori or a posteriori sensor parameters,
- the output of some observational process such as surveying.

First consider the case where only $\dot{\Delta}$ and $Q_{\dot{\Delta}\dot{\Delta}}$ are used by the production organization. This means that the organization has either the capability of measuring conjugate image coordinates from a stereo pair and then computing the ground coordinates or the capability of establishing the stereo model on some real time instrument and observing the model coordinates directly. Let G denote the ground positions output from this process, that is

$$G = F(\dot{\Delta}, \bar{L}) \qquad (III.1)$$

where $\bar{L}$ is the vector of either image or model observations. The function F is usually the projective relationship between the image coordinates and the model coordinates and/or some set of coordinate transformations of the observed position to the desired product reference system. The covariance matrix associated with G, denoted by Q_{GG}, is given by

$$Q_{GG} = \left[\partial F/\partial \dot{\Delta} \quad \partial F/\partial \bar{L}\right] \begin{bmatrix} Q_{\dot{\Delta}\dot{\Delta}} & 0 \\ 0 & Q_{\bar{L}\bar{L}} \end{bmatrix} \begin{bmatrix} (\partial F/\partial \dot{\Delta})^T \\ (\partial F/\partial \bar{L})^T \end{bmatrix}, \qquad (III.2)$$

where $Q_{\bar{L}\bar{L}}$ denotes the covariance matrix associated with $\bar{L}$ and must be determined during the observational process as an integral part of that process.

The vector G may consist of the coordinates of many ground positions, but without loss of generality assume that G is comprised of any two ground positions, each with three components, thus Q_{GG} is the 6 x 6 covariance matrix required to compute the relative circular and linear uncertainties, that is, if G is the product being evaluated, then

$$Q = Q_{GG}. \tag{III.3}$$

It is possible that ground coordinates derived from triangulated parameters may be provided to the production organization for use in the generation of a product or for their use as diagnostic control points for the purpose of product evaluation. If the ground points are used in the generation of a product, then the relationship of those ground positions, the state vector and subsequent observations, denoted by $\bar{L}$, required for product generation can be written functionally as

$$G' = F'(\dot{\Delta}, \bar{L}, G \text{ or } \ddot{\Delta}). \tag{III.4}$$

The function F' may consist of the function F as previously defined and some additional function describing a process involving the adjustment of observations to the vector G. The covariance matrix associated with G' is

$$Q_{G'G'} = \begin{bmatrix} \frac{\partial F'}{\partial \dot{\Delta}} & \frac{\partial F'}{\partial \bar{L}} & \frac{\partial F'}{\partial G} \end{bmatrix} \begin{bmatrix} Q_{\dot{\Delta}\dot{\Delta}} & 0 & 0 \\ 0 & Q_{\bar{L}\bar{L}} & 0 \\ 0 & 0 & Q_{GG} \end{bmatrix} \begin{bmatrix} (\partial F'/\partial \dot{\Delta})^T \\ (\partial F'/\partial \bar{L})^T \\ (\partial F'/\partial G)^T \end{bmatrix}. \tag{III.5}$$

Again $Q_{G'G'}$ can be computed for any two points and assuming that G' is the product output gives

$$Q = Q_{G'G'} \tag{III.6}$$

This concludes the discussion of error propagation with covariance matrices as a technique for product evaluation. Evaluation schemes based on this techniques require accurate knowledge of the a priori covariances associated with the state vector as well as accurate covariances associated with all observational processes involved with the generation of the product.

IV. Error Propagation from Sample Statistics

This section considers the task of product evaluation using sample statistics associated with residuals arising from comparisons of the product to diagnostic control points.

The following analysis is not dependent on the coordinate system definition, however eventually the resulting covariance matrix must be transformed to geographics or some local system with horizontal and vertical components.

Let the true, but unknown, coordinates of the j^{th} diagnostic point be denoted by $P_{T,j}$; let the coordinates of the same point as extracted from the product be denoted by P_j and let the coordinates of the point as used for diagnostic control be denoted by $P_{d,j}$.

The error sources contributing to the product evaluation process which will be considered during the following analysis are;

- biases in the product,
- random errors in the product,
- random errors in the diagnostic control positions and
- random errors in the measurements of the diagnostic positions in the product.

Let a residual vector, denoted by V_j , associated with the j^{th} diagnostic position extracted from the product be defined such that

$$P_{T,j} = P_j + V_j \quad , \tag{IV.1}$$

that is, the true position is equal to the product plus the residual vector. Likewise define a residual vector, denoted by $V_{d,j}$, associated with the diagnostic control position such that

$$P_{T,j} = P_{d,j} + V_{d,j} \, . \tag{IV.2}$$

These two residual vectors are related by

$$P_j - P_{d,j} = V_{d,j} - V_j \quad , \tag{IV.3}$$

thus the difference in coordinates of the points extracted from the product and the corresponding diagnostic control position is equal to the difference in their residual vectors. This fact is the key to product evaluation using sample statistics.

Define

$$\delta_j = V_{d,j} - V_j \,. \qquad (\text{IV}.4)$$

Assume that the diagnostic control positions are unbiased with respect to the product datum, that is

$$E[V_{d,j}] = 0. \qquad (\text{IV}.5)$$

This assumption is usually necessary because any bias in the diagnostic control positions are unknown and/or unmeasurable. Frequently the diagnostic control positions are generated with source from a population which has an unbiased error distribution. The covariance matrix associated with the source is usually applicable to the entire population or at least that part of the population with the same characteristics as the source used for the diagnostic derivation. This means that any biases in the diagnostic positions should be interpreted as an error from an unbiased population. Thus the associated covariance matrix is representative of all diagnostic points more so than those developed for the evaluation of a single product area.

Consider two points, point i and j, and let Q_{dd} denote the covariance matrix associated with the diagnostic control position such that it is partitioned as

$$Q_{dd} = \begin{bmatrix} Q_{d,i,i} & Q_{d,i,j} \\ Q^T_{d,i,j} & Q_{d,j,j} \end{bmatrix} \qquad (\text{IV}.6)$$

or

$$Q_{dd} = \begin{bmatrix} E[V_{d,i} V^T_{d,i}] & E[V_{d,i} V^T_{d,j}] \\ E[V_{d,j} V^T_{d,i}] & E[V_{d,j} V^T_{d,j}] \end{bmatrix}. \qquad (\text{IV}.7)$$

Thus Q_{dd} is a 6 x 6 covariance matrix with $Q_{d,k,k}$, k = i or j, being the 3 x 3 covariance matrix associated with the k^{th} diagnostic control position. In actual practice the total $Q_{d,d}$ matrix is seldom available. At best only the diagonal partitions are supplied. A more likely event is that a set of variances or standard deviations are provided that apply equally to all the points.

The mean residual vector, $\bar{V}$, of the diagnostic positions extracted from the product is defined as

$$\bar{V} = E[V_j] . \qquad (IV.8)$$

If $\bar{V} = 0$, then the product is unbiased. However this is not always the case and is not assumed. It is assumed that the residual vector associated with the j^{th} measured diagnostic position can be represented as the sum of the mean residual vector and an unbiased randomly distributed residual vector, denoted by $V_{R,j}$, that is

$$V_j = V_{R,j} + \bar{V} . \qquad (IV.9)$$

If systematic errors, other than biases, are detected in the product, then $\bar{V}$ should be replaced by the function defining that error, although it would be more appropriate to remove the effects of that systematic error from the product. Since

$$\delta_j = V_{d,j} - V_j \qquad (IV.10)$$

substitution for V_j gives

$$\delta_j = V_{d,j} - V_{R,j} - \bar{V} . \qquad (IV.11)$$

The mean value of all δ_j, denoted by $\bar{\delta}$, is

$$\bar{\delta} = E[\delta_j] \qquad (IV.12)$$

and since the residuals $V_{d,j}$ and $V_{R,j}$ are assumed to be samples from an unbiased population

$$\bar{\delta} = -\bar{V} . \qquad (IV.13)$$

The error source yet to be considered is that which occurs in the measurement of the diagnostic positions depicted by the product. Let the residual vector associated with the measurement of the j^{th} point be denoted by $V_{M,j}$ and defined such that

$$V_{R,j} = V_{P,j} + V_{M,j} , \qquad (IV.14)$$

where $V_{P,j}$ is that part of the random residual vector arising from the errors in the product. Both $V_{M,j}$ and $V_{P,j}$ are assumed to be samples from an unbiased normally distributed population. Therefore when all error sources are considered,

$$\delta_j = V_{d,j} - V_{P,j} - V_{M,j} - \bar{V} . \qquad (IV.15)$$

The covariance matrix associated with the random error of the two points i and j as depicted in the product is defined as

$$Q_{PP} = \begin{bmatrix} Q_{P,i,i} & Q_{P,i,j} \\ Q^T_{P,i,j} & Q_{P,j,j} \end{bmatrix} , \qquad (IV.16)$$

where

$$Q_{P,i,j} = E[V_{P,i} V^T_{P,j}] . \qquad (IV.16a)$$

Define

$$\delta = \begin{bmatrix} \delta_i \\ \delta_j \end{bmatrix} . \qquad (IV.17)$$

The covariance matrix of δ, denoted by $Q_{\delta\delta}$, is

$$Q_{\delta\delta} = E[(\delta - \bar{\delta})(\delta - \bar{\delta})^T] , \qquad (IV.18)$$

which simplifies to

$$Q_{\delta\delta} = E[\delta\delta^T] - \bar{V}\bar{V}^T . \qquad (IV.18a)$$

Substitution of the definition of δ and simplification, assuming the measurement errors are independent of all other errors, yields

$$Q_{\delta\delta} = Q_{dd} + Q_{PP} + Q_{MM} - Q_{dP} - Q_{Pd} \qquad (IV.19)$$

where

$$Q_{dP} = E[V_{d,j} V_{P,j}^T] \qquad (IV.20)$$

and

$$Q_{MM} = E[V_{M,j} V_{M,j}^T]. \qquad (IV.21)$$

The desired matrix Q giving the statistics associated with the product is

$$Q = Q_{PP} + B(\bar{V}), \qquad (IV.22)$$

where $B(\bar{V})$ is a matrix that is a function of the bias. The last section of this report will describe a methodology of computing CE and LE when affected by a bias. For the remainder of this section the bias term will not be considered, only the random errors are propagated. Therefore

$$Q = Q_{\delta\delta} - Q_{dd} - Q_{MM} + Q_{dP} + Q_{dP}^T. \qquad (IV.23)$$

This is a generalized form of the relationship between the covariance matrix associated with the product and those associated with the error sources in that the assumption was that of independence of the errors incurred with the measurement of the diagnostic points in the product. This equation is useful since it can be simplified according to the statistical relationship between the diagnostic control errors and the product errors.

IVa. Sample Statistics When the Diagnostic and Product Errors are Independent.

The first example to be considered is that where the error sources associated with the diagnostic control uncertainty and those associated with the product uncertainty are totally independent, that is,

$$Q_{dP} = 0 \qquad (IV.24)$$

thus

$$Q = Q_{\delta\delta} - Q_{dd} - Q_{MM}. \qquad (IV.25)$$

This situation frequently occurs in product source and evaluation where the diagnostic control results from a ground survey.

IVb. Sample Statistics When Diagnostic and Product Errors are Dependent.

The other situation to be considered is when the diagnostic control is developed from the same source as the product. In this instance the matrix

$$Q_{dP} \approx Q_{dd}, \tag{IV.26}$$

the only difference being the error associated with the measurement procedures for control development and those associated with product generation. Let the covariance matrix Q_{dd} be redefined as the sum of the covariance matrix associated with the measurements required for control development, denoted by $Q_{M'M'}$, and the covariance matrix associated with all other error sources related to control development, denoted by $Q_{d'd'}$, that is,

$$Q_{dd} = Q_{d'd'}. \tag{IV.27}$$

Since

$$Q_{dP} = Q_{d'd'} \tag{IV.28}$$

approximately, then

$$Q = Q_{\delta\delta} - Q_{d'd'} - Q_{M'M'} - Q_{MM} + Q_{d'd'} + Q_{d'd'} \tag{IV.29}$$

which simplifies to

$$Q = Q_{\delta\delta} + Q_{d'd'} - Q_{M'M'} - Q_{MM}. \tag{IV.30}$$

This appears to be a larger uncertainty than in the previous example in the sense that $Q_{d'd'}$ is added instead of subtracted. However, consider that

$$Q_{\delta\delta} = Q_{MM} + Q_{M'M'} + Q_{CC} \tag{IV.31}$$

approximately, where Q_{CC} is the covariance matrix associated with data extraction and subsequent processing required for product output. This means that all error sources in the diagnostic control and the product are the same except for the measurement errors, then

$$Q = Q_{d'd'} + Q_{CC} \tag{IV.32}$$

approximately, thus the uncertainty is not necessarily larger than in the previous example. This second example is the usual situation existing in the generation and evaluation of a product.

IVc. Summary of Sample Statistics Methodology.

To summarize, the following steps are required to compute the covariance matrix Q using sample statistics resulting from the comparison of the product to diagnostic control positions.

1. The production organization is supplied with diagnostic control positions which must be distributed such that a reasonable number of pairs for the point to point accuracy computations are available. The point to point relative accuracy is that accuracy associated with the horizontal and vertical components of a vector from one product point to another product point. Since the point to point accuracies are likely to be a function of the vector length the vectors can be classified according to length and an accuracy computed for each class. The accuracy assigned to the product is that associated with the class of vectors of length specified for the product. For the purposes of this report consider only a single class of vectors. Let the subscript i denote either the initial or terminal point of the j^{th} vector, that is, $P_{d,i,j}$ denotes the position of one of the points defining the j^{th} vector. The production organization should also be supplied with the covariance matrices $Q_{M'M'}$, Q_{dd} and/or $Q_{d'd'}$. It is possible that each diagnostic control position has a different covariance matrix supplied, but this is not the usual case. Usually a covariance matrix or standard deviations are supplied that apply equally to any point.

2. The production organization identifies the diagnostic locations in the product and measures their positions. These measurements are used to computed the geographics of the points, denoted by $P_{i,j}$.

3. The 6 x 6 cross-covariance matrix $Q_{\delta\delta}$ is computed by

$$Q_{\delta\delta} = \begin{bmatrix} E[\delta_1 \delta_1^T] & E[\delta_1 \delta_2^T] \\ E[\delta_2 \delta_1^T] & E[\delta_2 \delta_2^T] \end{bmatrix} - \bar{V}\bar{V}^T \quad (IV.33)$$

where

$$E[\delta_i \delta_i^T] = \frac{1}{m-1} \sum_{j=1}^{m} (P_{i,j} - P_{d,i,j})(P_{i,j} - P_{d,i,j})^T, \quad (IV.34)$$

$$E[\delta_i \delta_k^T] = \frac{1}{m-1} \sum_{j=1} (P_{i,j} - P_{d,i,j})(P_{k,j} - P_{d,k,j})^T \quad (IV.35)$$

and m is the number of pairs selected for the evaluation process. Since

the point to point accuracies assigned to a given product are for a specified distance, the diagnostic points must be paired so that they are separated by approximately that distance.

4. The covariance matrix Q_{MM} is determined. This will require experimentation using redundant observations. It is unlikely that this experiment needs to be repeated everytime a product area is evaluated.

5. The covariance matrix Q is computed using the appropriate equation,

$$Q = Q_{\delta\delta} - Q_{dd} - Q_{MM} \qquad (IV.36)$$

if the diagnostic control positions are derived from a source other than that used for the product generation, or

$$Q = Q_{\delta\delta} + Q_{d'd'} - Q_{M'M'} - Q_{MM} \qquad (IV.37)$$

if the diagnostic control and the product are developed from the same source.

V Absolute Accuracy Computations

The methodology for computing absolute and relative accuracy from the covariance matrix Q is now presented, where Q is partitioned such that

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} . \qquad (V.1)$$

Depending on the distribution and pairing of the diagnostic points, the covariance matrix Q_{11} can be associated with the absolute accuracy of a point located in a specific area of the total product area and the covariance matrix Q_{22} can be associated with an area at the distance required to satisfy the point to point accuracy specifications.

Regardless of the method used to determine Q it is likely that the angular units are radians and the linear units are meters. The proposed method of computing accuracy assumes that all units are meters.

For the purpose of this conversion a spherical earth, for each local area, is assumed with the radius defined by

$$R = a/(1 - e^2 SIN^2\phi)^{1/2} \qquad (V.2)$$

where a is the semi- major axis and e is the eccentricity of the reference ellipsoid. Any distance between two points, denoted by S, along this meridian can be approximated by

$$S = R\Delta\phi, \qquad (V.3)$$

where $\Delta\phi$ is the angular separation in latitude of the two points. Let $\hat{\sigma}_\phi$ and $\hat{\sigma}_\lambda$ denote the standard deviation of the latitude and longitude, respectively, in units of meters, thus

$$\hat{\sigma}_\phi = R\,\sigma_\phi \qquad (V.4)$$

and

$$\hat{\sigma}_\lambda = R\cos\phi\cdot\sigma_\lambda \qquad (V.5)$$

where σ_ϕ and σ_λ are the standard deviations of the latitude and longitude, respectively, in radians as extracted from Q_{11} or Q_{22}. The elements of the covariance matrix Q are modified such that the correlation between the variables is maintained. For example, let $\rho_{\phi\lambda}$ denote the correlation coefficient between ϕ and λ, thus

$$\rho_{\phi\lambda} = \sigma_{\phi\lambda}/\sigma_\phi\sigma_\lambda = \hat{\sigma}_{\phi\lambda}/\hat{\sigma}_\phi\hat{\sigma}_\lambda \qquad (V.6)$$

which is simplified to

$$\hat{\sigma}_{\phi\lambda} = R^2\cos\phi\cdot\sigma_{\phi\lambda}. \qquad (V.7)$$

Proceeding in this manner the units of all elements of Q can be converted to meters squared without changing the statistical content of the matrix. Let $\hat{Q}$ denoted the modified covariance matrix such that

$$\hat{Q} = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{12}^T & \hat{Q}_{22} \end{bmatrix}. \qquad (V.8)$$

The absolute CE and LE can be computed using either $\hat{Q}_{11}$ or $\hat{Q}_{22}$. If the product area being evaluated is of uniform accuracy the results should be essentially the same regardless of the choice. It is suggested that the absolute CE and LE be computed twice, once using $\hat{Q}_{11}$ and once using $\hat{Q}_{22}$. The maximum CE and LE should be selected for the product evaluation.

Consider either the covariance matrix Q_{11} or Q_{22}. Assume that the horizontal and vertical components of the uncertainty are independent. This assumption means that the covariances between the horizontal components and the vertical component are zero and Q_{11} is given by

$$C_{11} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} & 0 \\ \sigma_{\phi\lambda} & \sigma_\lambda^2 & 0 \\ 0 & 0 & \sigma_h^2 \end{bmatrix} . \qquad (V.9)$$

The subscripts on the matrix elements have been dropped since there is not a need to distinguish between the two partitions. If the covariances indicated are not near zero the axis of the error ellipsoid does not point in the vertical direction. Not much can be done about this condition unless the definitions of accuracies associated with the products are to be modified. Fortunatally these covariances are usually near zero.

The absolute LE is given by

$$LE = 1.6449 \left| \sqrt{\sigma_h^2} \right| . \qquad (V.10)$$

The procedure for determining the absolute CE is more complicated. To insure that the axes of the horizontal error ellipsoid are used in the computations it is necessary to find the eigenvalues of the horizontal partition of the covariance matrix. These eigenvalues are the magnitude of the axes of the horizontal error ellipsoid, thus

$$\begin{vmatrix} \sigma_\phi^2 - \mu & \sigma_{\phi\lambda} \\ \sigma_{\phi\lambda} & \sigma_\lambda^2 - \mu \end{vmatrix} = 0 \qquad (V.11)$$

gives the eigenvalues μ , which are

$$\mu = \frac{1}{2}(\sigma_\phi^2 + \sigma_\lambda^2) \pm \frac{1}{2}\sqrt{(\sigma_\phi^2 - \sigma_\lambda^2)^2 + 4(\sigma_{\phi\lambda})^2} . \qquad (V.12)$$

Note that if the two components are independent, then $\sigma_{\phi\lambda}=0$ and

$$\mu = \frac{1}{2}(\sigma_\phi^2 + \sigma_\lambda^2) \pm \frac{1}{2}(\sigma_\phi^2 - \sigma_\lambda^2) \qquad (V.13)$$

or

$$\mu = \sigma_\phi^2 \text{ or } \sigma_\lambda^2 \qquad (V.14)$$

as expected.

Let U and V denote the two independent variables whose uncertainty is given by the eigenvalues, that is

$$\sigma_U^2 = \frac{1}{2}(\sigma_\phi^2 + \sigma_\lambda^2) + \sqrt{\frac{1}{4}(\sigma_\phi^2 - \sigma_\lambda^2)^2 + (\sigma_{\phi\lambda})^2} \qquad (V.15)$$

and

$$\sigma_V^2 = \frac{1}{2}(\sigma_\phi^2 + \sigma_\lambda^2) - \sqrt{\frac{1}{4}(\sigma_\phi^2 - \sigma_\lambda^2)^2 + (\sigma_{\phi\lambda})^2}. \qquad (V.16)$$

The absolute CE is computed utilizing σ_U and σ_V. The usual method is to let

$$CE = 2.146(\sigma_U + \sigma_V)/2 \qquad (V.17)$$

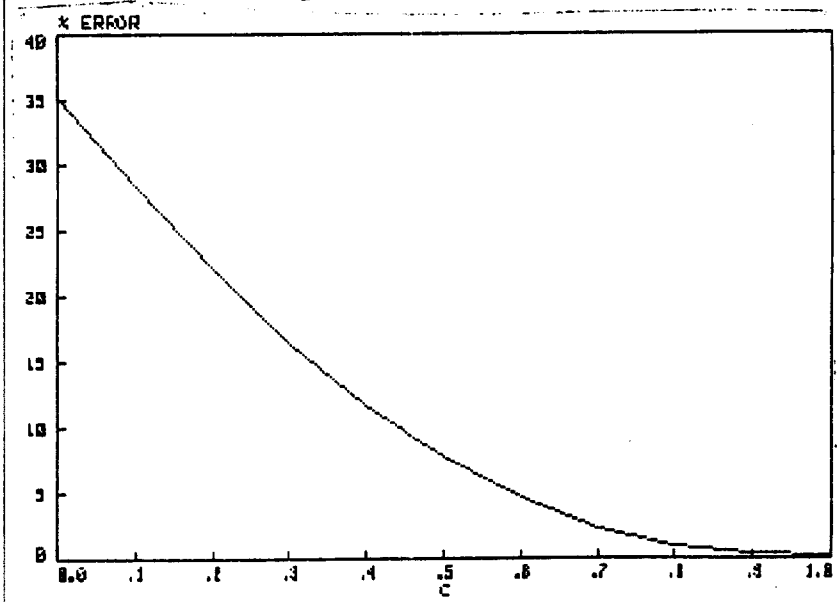
however this approximation is valid only if $0.5 \leq \sigma_V/\sigma_U \leq 1.0$. Let

$$C = \sigma_V/\sigma_U \qquad (V.18)$$

where

$$\sigma_V \leq \sigma_U. \qquad (V.19)$$

The graph illustrates the percent error for CE as a function of C when the absolute error is computed using the above approximation.

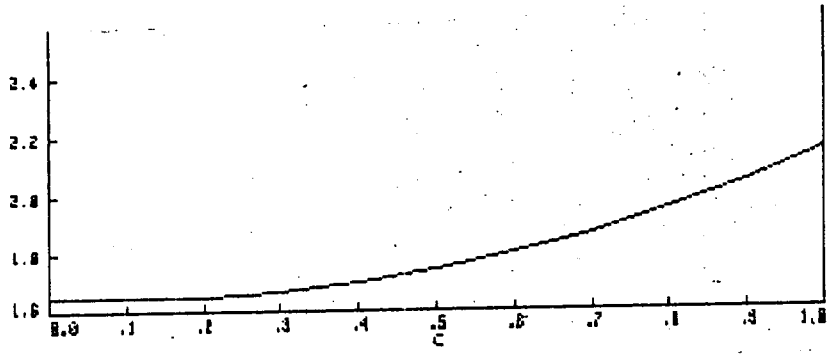


The error is such that the approximation gives results that are overly optimistic. The following method is suggested as a way to avoid this error.

Suppose that

$$CE = K\sigma_U . \qquad (V.20)$$

The following graph shows the correct value of K as a function of C .



This graph was constructed from data available in the CRC Handbook for Probability and Statistics, Second Edition, and is valid only if U and V are independent variables, which is insured since σ_U^2 and σ_V^2 are the eigenvalues of the covariance matrix.

A second order polynomial was fit to the values shown in the graph to obtain

$$K = 1.6545 - 0.13913C + 0.6324C^2 \qquad (V.21)$$

with

$$\sigma_K = 0.005 \qquad (V.22)$$

thus

$$CE = (1.6545 - 0.13913C + 0.6324C^2)\,\sigma_U \qquad (V.23)$$

VI. Point to Point Relative Accuracy Computations

Since the elements of the covariance matrix Q are in units of meters squared it is legitimate to use the components of the vector defined by the two points for the purpose of point to point accuracy computations. Let these components be denoted by

$$\Delta\phi = \phi_2 - \phi_1 \,, \tag{VI.1}$$

$$\Delta\lambda = \lambda_2 - \lambda_1 \tag{VI.2}$$

and

$$\Delta h = h_2 - h_1 \,. \tag{VI.3}$$

The objective is to determine the covariance matrix associated with $\Delta\phi$, $\Delta\lambda$ and Δh. This covariance matrix is then used to compute the relative point to point CE and LE. Let this covariance matrix be denoted by Q_{pp}, thus

$$Q_{pp} = J Q J^T \tag{VI.4}$$

where

$$J = \begin{bmatrix} \frac{\partial\Delta\phi}{\partial\phi_1} & \frac{\partial\Delta\phi}{\partial\lambda_1} & \frac{\partial\Delta\phi}{\partial h_1} & \frac{\partial\Delta\phi}{\partial\phi_2} & \frac{\partial\Delta\phi}{\partial\lambda_2} & \frac{\partial\Delta\phi}{\partial h_2} \\ \frac{\partial\Delta\lambda}{\partial\phi_1} & \frac{\partial\Delta\lambda}{\partial\lambda_1} & \frac{\partial\Delta\lambda}{\partial h_1} & \frac{\partial\Delta\lambda}{\partial\phi_2} & \frac{\partial\Delta\lambda}{\partial\lambda_2} & \frac{\partial\Delta\lambda}{\partial h_2} \\ \frac{\partial\Delta h}{\partial\phi_1} & \frac{\partial\Delta h}{\partial\lambda_1} & \frac{\partial\Delta h}{\partial h_1} & \frac{\partial\Delta h}{\partial\phi_2} & \frac{\partial\Delta h}{\partial\lambda_2} & \frac{\partial\Delta h}{\partial h_2} \end{bmatrix} \,, \tag{VI.5}$$

thus

$$J = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \,. \tag{VI.6}$$

It is assumed that the horizontal and vertical differences are independent, thus the covariance matrix Q_{pp} is of the form

$$Q_{pp} = \begin{bmatrix} \sigma^2_{\Delta\phi} & \sigma_{\Delta\phi\Delta\lambda} & 0 \\ \sigma_{\Delta\phi\Delta\lambda} & \sigma^2_{\Delta\lambda} & 0 \\ 0 & 0 & \sigma^2_{\Delta h} \end{bmatrix} \,. \tag{VI.7}$$

Let

$$Q_{PP} = \begin{bmatrix} \hat{Q}_{PP} & 0 \\ 0 & \sigma^2_{\Delta h} \end{bmatrix}, \qquad (VI.8)$$

that is

$$\hat{Q}_{PP} = \begin{bmatrix} \sigma^2_{\Delta\phi} & \sigma_{\Delta\phi\Delta\lambda} \\ \sigma_{\Delta\phi\Delta\lambda} & \sigma^2_{\Delta\lambda} \end{bmatrix}. \qquad (VI.9)$$

If $\sigma_{\Delta\phi\Delta\lambda}=0$, then $\Delta\phi$ and $\Delta\lambda$ are independent variables, but if $\sigma_{\Delta\phi\Delta\lambda}\neq 0$ it is necessary to determine the semi-major and semi-minor axes of the error ellipse defined by $\hat{Q}_{PP}$. The length of these axes are the standard deviations of the two independent variables, say U and V. The variances for U and V are the eigenvalues of $\hat{Q}_{PP}$, thus

$$\sigma^2_U = \frac{1}{2}(\sigma^2_{\Delta\phi} + \sigma^2_{\Delta\lambda}) + \sqrt{\frac{1}{4}(\sigma^2_{\Delta\phi} - \sigma^2_{\Delta\lambda})^2 + (\sigma_{\Delta\phi\Delta\lambda})^2} \qquad (VI.10)$$

and

$$\sigma^2_V = \frac{1}{2}(\sigma^2_{\Delta\phi} + \sigma^2_{\Delta\lambda}) - \sqrt{\frac{1}{4}(\sigma^2_{\Delta\phi} - \sigma^2_{\Delta\lambda})^2 + (\sigma_{\Delta\phi\Delta\lambda})^2}. \qquad (VI.11)$$

The remaining steps for computing relative CE and LE from σ_U and σ_V are identical to those described for computing absolute accuracy.

VII Alternate Error Propagation from Sample Statistics

An alternate method of obtaining the product accuracies from sample statistics will now be described. This method requires fewer computations, but is recommended for use only when the diagnostic control uncertainty and the measurement uncertainty associated with the extraction of the diagnostic points from the product is small when compared to the uncertainty associated with the product.

Suppose that the diagnostic control positions are in terms of geographics, that is the points (ϕ_j, λ_j, h_j), $j = 1, 2, \ldots$ are provided to the production organization. Let the corresponding positions as measured in the product be denoted by $(\hat{\phi}_j, \hat{\lambda}_j, \hat{h}_j)$. The difference of the

product and diagnostic control for the j^{th} point is

$$\begin{bmatrix} \Delta\phi_j \\ \Delta\lambda_j \\ \Delta h_j \end{bmatrix} = \begin{bmatrix} \hat{\phi}_j - \phi_j \\ \hat{\lambda}_j - \lambda_j \\ \hat{h}_j - h_j \end{bmatrix} . \qquad (VII.1)$$

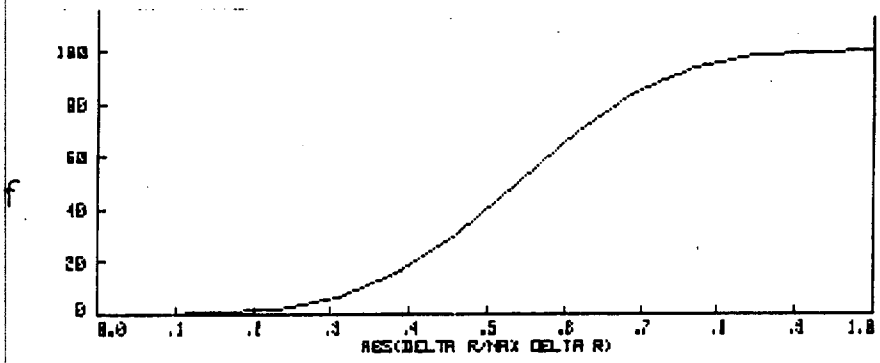
If the assumptions of negligible diagnostic control uncertainty and measurement uncertainty are valid these differences represent the errors in the product at that point.

If the units associated with these differences are not in meters they should be converted as previously detailed.

Let ΔR_j denote the horizontal error associated with the j^{th} diagnostic point, that is

$$\Delta R_j = \sqrt{\Delta\phi^2 + \Delta\lambda^2} . \qquad (VII.2)$$

These horizontal errors are used to construct an ogive. An ogive is defined as a graph showing the cumulative frequency less than ΔR plotted against ΔR . The following graph illustrates the concept of an ogive.



In practice the basic idea of the method is to delete the points with the largest differences until 10 percent of the samples have been eliminated. The largest remaining residual is taken to be the absolute CE.

The absolute LE is determined in a like manner using

$$\Delta h_j = \hat{h}_j - h_j \qquad \text{(VII.3)}$$

and discarding the largest differences until 10 percent of the differences are eliminated. The largest remaining difference is the absolute LE. Any systematic error in the product will contribute to the accuracy when determined from an ogive, but an assessment of the magnitude is not obtained.

The relative point to point accuracy can also be determined by use of the ogive. Consider the vector, denoted by d_j , between the two diagnostic control positions. Let

$$d_j = \begin{bmatrix} d\phi_j \\ d\lambda_j \\ dh_j \end{bmatrix} \qquad \text{(VII.4)}$$

denote the components of this vector. The points should be separated by approximately the distance required by the product specification for point to point accuracy computations. Let $\hat{d}j$ denote the vector between the same two points as depicted in the product, that is

$$\hat{d}_j = \begin{bmatrix} d\hat{\phi}_j \\ d\hat{\lambda}_j \\ d\hat{h}_j \end{bmatrix}. \qquad \text{(VII.5)}$$

Define for each diagnostic pair,

$$\Delta d_j = \hat{d}_j - d_j = \begin{bmatrix} d\hat{\phi}_j - d\phi_j \\ d\hat{\lambda}_j - d\lambda_j \\ d\hat{h}_j - dh_j \end{bmatrix} \qquad \text{(VII.6)}$$

and assume that all units are meters. Let

$$\Delta r_j = \sqrt{(d\hat{\phi}_j - d\phi)^2 + (d\hat{\lambda}_j - d\lambda_j)^2} \qquad \text{(VII.7)}$$

and form the ogive to obtain the relative CE.

Likewise form the ogive using

$$\Delta h_j = \left| d\hat{h}_j - dh_j \right| \qquad \text{(VII.8)}$$

and determine the relative LE.

VIII. Accuracy Influenced by Bias.

The affects of bias on the evaluation process is now considered. This section details the methodology for the incorporation of the bias into the accuracy computations. While the development presented is in terms of absolute accuracies, the same relationships are applicable to relative accuracy computations.

The assumptions of section IV are still valid, in particular the assumption that the product, not the diagnostic control, is biased. It is further assumed that the sample statistics arising from the evaluation process described in section IV are captured in the form of the covariance matrix Q (equation IV.23) and the bias vector $\bar{V}$ (equation IV.8).

First consider the bias applied to the CE computations. The methodology described is based on a study report entitled CIRCULAR ERROR PROBABILITY OF A QUANTITY AFFECTED BY A BIAS, by Melvin E. Shultz, DMAAC.

Define the components of the bias vector V as

$$V = \begin{bmatrix} \bar{V}_\phi \\ \bar{V}_\lambda \\ \bar{V}_h \end{bmatrix}, \qquad \text{(VIII.1)}$$

then the horizontal bias, denoted by b_h, is given by

$$b_h = \sqrt{\bar{V}_\phi^2 + \bar{V}_\lambda^2} \qquad \text{(VIII.2)}$$

where b_h is always positive and is considered to be in units of meters.

The values of σ_U^2 and σ_V^2 as defined by equations V.15 and V.16 are extracted from the covariance matrix Q. The resulting values of σ_U and σ_V are used to compute CE.

The equations developed by Shultz require a .39 P circular error as input, thus define σ_c such that

$$\sigma_c = 0.4660\ CE. \qquad \text{(VIII.3)}$$

Without loss of generality assume that the bias is along the x axis of an arbitrary coordinate system. Also assume that the density function, which is that associated with a circular normal distribution, is centered at $(b_h, 0)$. From the definition of probability if follows that

$$P[X^2+Y^2<CE^2] = \int\int \frac{1}{2\pi\sigma_c^2} \exp\left\{\frac{-1}{2\sigma_c^2}\left[(x-b_h)^2+y^2\right]\right\} dx\,dy \qquad \text{(VIII.4)}$$

or in terms of polar coordinates

$$P[r^2<CE^2] = \exp\left\{\frac{-b_h^2}{2\sigma_c^2}\right\} \frac{1}{2\pi\sigma_c^2} \int_0^{CE}\int_0^{2\pi} \exp\left\{\frac{-1}{2\sigma_c^2}\left[r^2-2rb_h\cos\theta\right]\right\} r\,d\theta\,dr. \qquad \text{(VIII.5)}$$

This equation must be evaluated such that $P[r<CE]=.9$ is satisfied. The study report by Shultz provides an algorithm for evaluating the probability. In addition a table of circular probabilities as a function of b_h/σ_c and CE/σ_c is given in the CRC HANDBOOK OF TABLES FOR PROBABILITY AND STATISTICS, 2nd edition. These tabulated values can be used to verify the equation given by Shultz,

$$\frac{CE}{\sigma_c} = 2.1272 + 0.1674\left(\frac{b_h}{\sigma_c}\right) + 0.3623\left(\frac{b_h}{\sigma_c}\right)^2 - 0.0550\left(\frac{b_h}{\sigma_c}\right)^3. \qquad \text{(VIII.6)}$$

Thus the CE, with bias , is given by

$$CE = 2.1272\sigma_c + 0.1674\,b_h + 0.3623\frac{b_h^2}{\sigma_c} - 0.0550\frac{b_h^3}{\sigma_c^2}. \qquad \text{(VIII.7)}$$

The LE, when influenced by a bias, is computed in a similar manner.

The value of σ_h , the standard deviation (.69 P) associated with the height, is extracted from the covariance matrix Q. Let b_v denote the vertical bias such that

$$b_v = \bar{v}_h. \qquad \text{(VIII.8)}$$

It can be shown that

$$LE = |b_v| + K\sigma_h \qquad \text{(VIII.9)}$$

where K is a function of the magnitude of the bias and standard deviation. The value of K is defined by the following table which is extracted from normal distribution tables.

$\lvert b_v \rvert / \sigma_h$	K
0.0	1.6449
0.1	1.5527
0.2	1.4772
0.3	1.4176
0.4	1.3716
0.5	1.3389
0.6	1.3158
0.7	1.3016
0.8	1.2924
0.9	1.2875
1.0	1.2844
1.1	1.2829
1.2	1.2824
1.3	1.2821
1.4	1.2815

When $|b_v| < 1.4$, $K = 1.2815$.
The values of K could be computed from the equation

$$P[x<LE] = \frac{1}{\sqrt{2\pi}\sigma_h} \int_{-(K+2|b_v|)}^{K} \exp\left\{\frac{-1}{2\sigma_h^2}(x-|b_v|)^2\right\} dx \,, \qquad \text{(VIII.7)}$$

solving for K so that $P[x<LE] = .9$.

A cubic polynomial fit through the tabular values yields

$$K = 1.6435 - 0.999556\frac{|b_v|}{\sigma_h} + 0.923237\left(\frac{|b_v|}{\sigma_h}\right)^2 - 0.282533\left(\frac{|b_v|}{\sigma_h}\right)^3 .$$

This value of K should be used when $|b_v|/\sigma_h \leq 1.4$. (VIII.8)